

# $k$ sequences of Generalized Van der Laan and Generalized Perrin Polynomials

Kenan Kaygisiz<sup>a,\*</sup>, Adem Şahin<sup>a</sup>

<sup>a</sup>*Department of Mathematics, Faculty of Arts and Sciences, Gaziosmanpaşa University,  
60250 Tokat, Turkey*

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## Abstract

In this paper, we present  $k$  sequences of Generalized Van der Laan Polynomials and Generalized Perrin Polynomials using Generalized Fibonacci and Lucas Polynomials. We give some properties of these polynomials. We also obtain generalized order- $k$  Van der Laan Numbers,  $k$  sequences of generalized order- $k$  Van der Laan Numbers, generalized order- $k$  Perrin Numbers and  $k$  sequences of generalized order- $k$  Perrin Numbers. In addition, we examine the relationship between them.

*Keywords:* Padovan Numbers, Cordonnier Numbers, Generalized Van der Laan Polynomials, Generalized Perrin Polynomials,  $k$  sequences of the generalized Van der Laan and Perrin Polynomials.

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## 1. Introduction

Fibonacci, Lucas, Pell and Perrin numbers are known for a long time. There are a lot of studies, relations, and applications of them. Generalization of this numbers has been studied by many researchers.

Miles [11] defined generalized order- $k$  Fibonacci numbers (GO $k$ F) in 1960. Er [1] defined  $k$  sequences of generalized order- $k$  Fibonacci Numbers ( $k$ SO $k$ F) and gave matrix representation for this sequences in 1984. Kalman [2] obtained a Binet formula for these sequences in 1982. Karaduman [3], Taşçı and Kılıç [13] studied on these sequences. Kılıç and Taşçı [7] defined  $k$  sequences of generalized order- $k$  Pell Numbers ( $k$ SO $k$ P) and obtained sums properties by using matrix method. Kaygisiz and Bozkurt [5] studied on generalization of Perrin numbers. Yılmaz and Bozkurt [14] give some properties of Perrin and Pell numbers.

Meanwhile, MacHenry [8] defined generalized Fibonacci polynomials ( $F_{k,n}(t)$ ), Lucas polynomials ( $G_{k,n}(t)$ ) in 1999, studied on these polynomials in [9] and defined matrices  $A_{(k)}^\infty$  and  $D_{(k)}^\infty$  in [10]. These studies of MacHenry include most

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\*Corresponding author (+903562521616-3087)

Email addresses: [kenan.kaygisiz@gop.edu.tr](mailto:kenan.kaygisiz@gop.edu.tr) (Kenan Kaygisiz),  
[adem.sahin@gop.edu.tr](mailto:adem.sahin@gop.edu.tr) (Adem Şahin)

of other studies mentioned above. For example,  $A_{(k)}^\infty$  is reduced to  $k$  sequences of generalized order- $k$  Fibonacci Numbers and  $A_{(k)}^\infty$  is reduced to  $k$  sequences of generalized order- $k$  Pell Numbers when  $t_1 = 2$  and  $t_i = 1$  (for  $2 \leq i \leq k$ ). In addition Binet formulas for  $(kSOkF)$  and  $(kSOkP)$  can be obtained by using equation (7). This analogy shows the importance of the matrices  $A_{(k)}^\infty$  and  $D_{(k)}^\infty$  and Generalized Fibonacci and Lucas polynomials. Based on this idea Kaygisiz and Şahin defined  $k$  sequences of generalized order- $k$  Lucas Numbers using  $G_{k,n}(t)$  and  $D_{(k)}^\infty$  in [4].

In this article, we first present  $k$  sequences of Generalized Van der Laan and Perrin Polynomials ( $V_{k,n}^i(t)$  and  $R_{k,n}^i(t)$ ) using Generalized Fibonacci and Lucas Polynomials and obtain generalized order- $k$  Van der Laan and Perrin Numbers,  $k$  sequences of generalized order- $k$  Van der Laan and Perrin Numbers by the help of these polynomials and matrices  $A_{(k)}^\infty$  and  $D_{(k)}^\infty$ . In addition, we examine the relationship between them and explore some of the properties of these sequences. We believe that our result are important, especially, for those who are interested in well known Fibonacci, Lucas, Pell and Perrin sequences and their generalization.

MacHenry [8] defined generalized Fibonacci polynomials ( $F_{k,n}(t)$ ), Lucas polynomials ( $G_{k,n}(t)$ ) and obtained important relations between generalized Fibonacci and Lucas polynomials in [9], where  $t_i$  ( $1 \leq i \leq k$ ) are constant coefficients of the core polynomial

$$P(x; t_1, t_2, \dots, t_k) = x^k - t_1 x^{k-1} - \dots - t_k, \quad (1)$$

which is denoted by the vector

$$t = (t_1, t_2, \dots, t_k). \quad (2)$$

$F_{k,n}(t)$  is defined inductively by

$$\begin{aligned} F_{k,n}(t) &= 0, \quad n < 0 \\ F_{k,0}(t) &= 1 \\ F_{k,1}(t) &= t_1 \\ F_{k,n+1}(t) &= t_1 F_{k,n}(t) + \dots + t_k F_{k,n-k+1}(t). \end{aligned}$$

$G_{k,n}(t_1, t_2, \dots, t_k)$  is defined by

$$\begin{aligned} G_{k,n}(t) &= 0, \quad n < 0 \\ G_{k,0}(t) &= k \\ G_{k,1}(t) &= t_1 \\ G_{k,n+1}(t) &= t_1 G_{k,n}(t) + \dots + t_k G_{k,n-k+1}(t). \end{aligned}$$

In [10], matrices  $A_{(k)}^\infty$  and  $D_{(k)}^\infty$  are defined by using the following matrix,

$$A_{(k)} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ t_k & t_{k-1} & t_{k-2} & \dots & t_1 \end{bmatrix}. \quad (3)$$

$A_{(k)}^\infty$  is obtained by multiplying  $A_{(k)}$  and  $A_{(k)}^{-1}$  by the vector  $t$  in (2). For  $k = 3$ ,  $A_{(k)}^\infty$  looks like this

$$A_{(3)}^\infty = \begin{bmatrix} \dots & \dots & \dots \\ S_{(-n,1^2)} & -S_{(-n,1)} & S_{(-n)} \\ \dots & \dots & \dots \\ S_{(-3,1^2)} & -S_{(-3,1)} & S_{(-3)} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ t_3 & t_2 & t_1 \\ \dots & \dots & \dots \\ S_{(n-1,1^2)} & -S_{(n-1,1)} & S_{(n-1)} \\ S_{(n,1^2)} & -S_{(n,1)} & S_{(n)} \\ \dots & \dots & \dots \end{bmatrix}$$

and

$$A_{(k)}^n = \begin{bmatrix} (-1)^{k-1}S_{(n-k+1,1^{k-1})} & \dots & (-1)^{k-j}S_{(n-k+1,1^{k-j})} & \dots & S_{(n-k+1)} \\ \dots & \dots & \dots & \dots & \dots \\ (-1)^{k-1}S_{(n,1^{k-1})} & \dots & (-1)^{k-j}S_{(n,1^{k-j})} & \dots & S_{(n)} \end{bmatrix} \quad (4)$$

where

$$S_{(n-r,1^r)} = (-1)^r \sum_{j=r+1}^n t_j S_{(n-j)}, \quad 0 \leq r \leq n. \quad (5)$$

Derivative of the core polynomial (1) is

$$P'(x) = kx^{k-1} - t_1(k-1)x^{k-2} - \dots - t_{k-1},$$

which is represented by the vector

$$(-t_{k-1}, \dots, -t_1(k-1), k). \quad (6)$$

Multiplying  $A_{(k)}$  and  $A_{(k)}^{-1}$  by the vector (6) gives the matrix  $D_{(k)}^\infty$ .

Right hand column of  $A_{(k)}^\infty$  contains sequence of the generalized Fibonacci polynomials  $F_{k,n}(t)$ . In addition, the right hand column of  $D_{(k)}^\infty$  contains sequence of the generalized Lucas polynomials  $G_{k,n}(t)$ .

For easier reference, we have state some theorems which will be used in the subsequent section. We also give some sequences mentioned above.

**Theorem 1.1.** [9]  $F_{k,n}(t)$  and  $G_{k,n}(t)$  are Generalized Fibonacci and Lucas polynomials respectively,

$$\sum_{j=1}^k \frac{\partial G_{k,n}(t)}{\partial t_j} t_j = nF_{k,n+1}(t)$$

and

$$nF_{k,n+1}(t) = \sum_{r=1}^k G_{k,r}(t)F_{k,n-r+1}(t).$$

**Theorem 1.2.** [9] Let  $\lambda_j$  be the roots of the polynomial (1) and let

$$\Delta_k = \begin{vmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_k \\ \vdots & & \vdots \\ \lambda_1^{k-1} & \cdots & \lambda_k^{k-1} \end{vmatrix} \text{ and } \Delta_{k,n} = \begin{vmatrix} 1 & \cdots & 1 \\ \lambda_1 & \cdots & \lambda_k \\ \vdots & & \vdots \\ \lambda_1^{k-2} & \cdots & \lambda_k^{k-2} \\ \lambda_1^{n+k-2} & \cdots & \lambda_k^{n+k-2} \end{vmatrix},$$

then we have

$$F_{k,n+1}(t) = \frac{\Delta_{k,n}}{\Delta_k}. \quad (7)$$

**Theorem 1.3.** [10]  $A_{(k)}$  is  $k \times k$  matrix in (3) then,

$$\det A_{(k)} = (-1)^{k+1} t_k$$

and

$$\det A_{(k)}^n = (-1)^{n(k+1)} t_k^n. \quad (8)$$

Miles [11] defined generalized order- $k$  Fibonacci numbers(GOkF) as,

$$f_{k,n} = \sum_{j=1}^k f_{k,n-j} \quad (9)$$

for  $n > k \geq 2$ , with boundary conditions:  $f_{k,1} = f_{k,2} = f_{k,3} = \cdots = f_{k,k-2} = 0$  and  $f_{k,k-1} = f_{k,k} = 1$ .

Er [1] defined  $k$  sequences of generalized order- $k$  Fibonacci Numbers( $k$ SOkF) as; for  $n > 0$ ,  $1 \leq i \leq k$

$$f_{k,n}^i = \sum_{j=1}^k c_j f_{k,n-j}^i \quad (10)$$

with boundary conditions for  $1 - k \leq n \leq 0$ ,

$$f_{k,n}^i = \begin{cases} 1 & \text{if } i = 1 - n, \\ 0 & \text{otherwise,} \end{cases}$$

where  $c_j$  ( $1 \leq j \leq k$ ) are constant coefficients,  $f_{k,n}^i$  is the  $n$ -th term of  $i$ -th sequence of order  $k$  generalization.  $k$ -th column of this generalization involves the Miles generalization for  $i = k$ , i.e.  $f_{k,n}^k = f_{k,k+n-2}$ .

Kılıç [7] defined  $k$  sequences of generalized order- $k$  Pell Numbers( $kSOkP$ ) as; for  $n > 0$ ,  $1 \leq i \leq k$

$$P_{k,n}^i = 2P_{k,n-1}^i + P_{k,n-2}^i + \cdots + P_{k,n-k}^i \quad (11)$$

with boundary conditions for  $1 - k \leq n \leq 0$ ,

$$P_{k,n}^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise} \end{cases}$$

where  $P_{k,n}^i$  is the  $n$ -th term of  $i$ -th sequence of order  $k$  generalization.

The well-known Cordonnier(Padovan) sequence  $\{C_n\}$  is defined recursively by the equation,

$$C_n = C_{n-2} + C_{n-3}, \text{ for } n > 3$$

where  $P_1 = 1$ ,  $P_2 = 1$ ,  $P_3 = 1$ .

Van der Laan sequence  $\{V_n\}$  is defined recursively by the equation,

$$V_n = V_{n-2} + V_{n-3}, \text{ for } n > 3$$

where  $V_1 = 1$ ,  $V_2 = 0$ ,  $V_3 = 1$ .

Perrin sequence  $\{R_n\}$  is defined recursively by the equation,

$$R_n = R_{n-2} + R_{n-3}, \text{ for } n > 3$$

where  $R_1 = 0$ ,  $R_2 = 2$ ,  $R_3 = 3$ [12].

In this paper we studied on generalized order- $k$  Van der Laan numbers  $v_{k,n}$  and  $k$  sequences of the generalized order- $k$  Van der Laan numbers  $v_{k,n}^i$  with the help of  $k$  sequences of Generalized Van der Laan Polynomials and generalized order- $k$  Perrin numbers  $r_{k,n}$  and  $k$  sequences of the generalized order- $k$  Perrin numbers  $r_{k,n}^i$  with the help of  $k$  sequences of Generalized Perrin Polynomials.

## 2. Generalized Van der Laan and Perrin Polynomials

Firstly we define generalized Van der Laan polynomial and  $k$  sequences of generalized Van der Laan polynomial by the help of generalized Fibonacci polynomials  $(F_{k,n}(t))$  and matrices  $A_{(k)}^\infty$ .

**Definition 2.1.** Generalized Fibonacci polynomials  $(F_{k,n}(t))$  is called generalized Van der Laan polynomials in case  $t_1 = 0$  for  $k \geq 3$ . So generalized Van der Laan polynomials are

$$\begin{aligned}
V_{k,0}(t) &= 0 \\
V_{k,1}(t) &= 1 \\
V_{k,2}(t) &= 0 \\
V_{k,3}(t) &= t_2 V_{k,1}(t) \\
V_{k,4}(t) &= t_2 V_{k,2}(t) + t_3 V_{k,1}(t) \\
&\vdots \\
V_{k,k-1}(t) &= t_2 V_{k,k-3}(t) + \cdots + t_{k-1} V_{k,1}(t)
\end{aligned}$$

and for  $n \geq k$

$$V_{k,n}(t) = \sum_{i=2}^k t_i V_{k,n-i}(t).$$

For  $k \geq 3$  substituting  $t_1 = 0$ , generalized Fibonacci polynomials  $(F_{k,n}(t))$  and matrices  $A_{(k)}^\infty$  are reduced to following polynomials; for  $n > 0$ ,  $1 \leq i \leq k$

$$V_{k,n}^i(t) = \sum_{j=2}^k t_j V_{k,n-j}^i(t) \quad (12)$$

with boundary conditions for  $1 - k \leq n \leq 0$ ,

$$V_{k,n}^i(t) = \begin{cases} 1 & \text{if } k = i - n, \\ 0 & \text{otherwise,} \end{cases}$$

where  $V_{k,n}^i(t)$  is the  $n$ -th term of  $i$ -th sequence of order  $k$  generalization.

**Definition 2.2.** The polynomials derived in (12) is called  $k$  sequences of generalized Van der Laan polynomials.

We note that for  $i = k$  and  $n \geq 0$ ,  $V_{k,n-1}^i(t) = V_{k,n}(t)$ .

**Example 2.3.** We give  $k$  sequences of generalized Van der Laan polynomial

$V_{k,n}^i(t)$  for  $k = 3$  and  $k = 4$

$n \setminus i$	1	2	3
-2	1	0	0
-1	0	1	0
0	0	0	1
1	$t_3$	$t_2$	0
2	0	$t_3$	$t_2$
3	$t_2 t_3$	$t_2^2$	$t_3$
4	$t_3^2$	$2t_2 t_3$	$t_2^2$
5	$t_2^2 t_3$	$t_2^3 + t_3^2$	$2t_2 t_3$
6	$2t_2 t_3^2$	$3t_2^2 t_3$	$t_2^3 + t_3^2$
7	$t_3^3 + t_2^3 t_3$	$t_2^4 + 3t_2^2 t_3^2$	$3t_2^2 t_3$
8	$\vdots$	$\vdots$	$\vdots$

$n \setminus i$	1	2	3	4
-3	1	0	0	0
-2	0	1	0	0
-1	0	0	1	0
0	0	0	0	1
1	$t_4$	$t_3$	$t_2$	0
2	0	$t_4$	$t_3$	$t_2$
3	$t_2 t_4$	$t_2 t_3$	$t_4 + t_2^2$	$t_3$
4	$t_3 t_4$	$t_2 t_4 + t_3^2$	$2t_2 t_3$	$t_4 + t_2^2$
5	$\vdots$	$\vdots$	$\vdots$	$\vdots$

In addition

$$V_{(k)} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ t_k & t_{k-1} & t_{k-2} & \dots & 0 \end{bmatrix}$$

is the generator matrix of  $k$  sequences of generalized Van der Laan polynomials. Matrix  $V_{(k)}^\infty$  is obtained by multiplying  $V_{(k)}$  and  $V_{(k)}^{-1}$  by the vector  $v = (t_k, t_{k-1}, t_{k-2}, \dots, 0)$ .

Note that it is also possible to obtain matrix  $V_{(k)}^\infty$  from matrix  $A_{(k)}^\infty$  by substituting  $t_1 = 0$ .

Let  $\widetilde{V}_n$  be generalized Van der Laan matrix which is obtained by  $n$ -th power of  $V_{(k)}$  as;

$$\widetilde{V}_n = (V_{(k)})^n = \begin{bmatrix} V_{k,n-k+1}^1(t) & V_{k,n-k+1}^2(t) & \dots & V_{k,n-k+1}^k(t) \\ \vdots & \vdots & \dots & \vdots \\ V_{k,n-1}^1(t) & V_{k,n-1}^2(t) & \ddots & V_{k,n-1}^k(t) \\ V_{k,n}^1(t) & V_{k,n}^2(t) & \dots & V_{k,n}^k(t) \end{bmatrix}. \quad (13)$$

It is obvious that  $V_{(k)} = \widetilde{V}_1$ .

**Corollary 2.4.** Let  $\widetilde{V}_n$  be as in (13). Then

$$\det \widetilde{V}_n = (-1)^{n(k+1)} t_k^n.$$

*Proof.* It is direct from Theorem 1.3. □

We define generalized Perrin polynomial and matrix  $R_{(k)}^\infty$  by the help of generalized Lucas polynomials  $(G_{k,n}(t))$  and matrices  $D_{(k)}^\infty$ .

**Definition 2.5.** Generalized Lucas polynomials  $(G_{k,n}(t))$  are called generalized Perrin polynomials in case  $t_1 = 0$  for  $k \geq 3$ . So generalized Perrin polynomials are;

$$\begin{aligned} R_{k,0}(t) &= k \\ R_{k,1}(t) &= 0 \\ R_{k,2}(t) &= 2t_2 \\ R_{k,3}(t) &= t_2 R_{k,1}(t) + 3t_3 \\ R_{k,4}(t) &= t_2 R_{k,2}(t) + t_3 R_{k,1}(t) + 4t_4 \\ &\vdots \\ R_{k,k-1}(t) &= t_2 R_{k,k-3}(t) + \cdots + t_{k-1} R_{k,1}(t) + kt_k \end{aligned}$$

and for  $n \geq k$ ,

$$R_{k,n}(t) = \sum_{i=2}^k t_i R_{k,n-i}(t).$$

We obtain matrix  $R_{(k)}$  by using row vector

$$(-t_{k-1}, \dots, -t_2(k-2), 0, k)$$

which is obtained from coefficient of derivative of core polynomial (1).  $k$ -th row of matrix  $R_{(k)}$  is the vector

$$(-t_{k-1}, \dots, -t_2(k-2), 0, k)$$

and  $i$ -th row is

$$(-t_{k-1}, \dots, -t_2(k-2), 0, k)(V_{(k)})^{-(k-i)}$$

for  $1 \leq i \leq k-1$ . It looks like

$$R_{(k)} = \begin{bmatrix} (-t_{k-1}, \dots, -t_2(k-2), 0, k) \cdot (V_{(k)})^{-(k-1)} \\ (-t_{k-1}, \dots, -t_2(k-2), 0, k) \cdot (V_{(k)})^{-(k-2)} \\ \vdots \\ (-t_{k-1}, \dots, -t_2(k-2), 0, k) \cdot (V_{(k)})^{-1} \\ (-t_{k-1}, \dots, -t_2(k-2), 0, k) \end{bmatrix}_{k \times k}. \quad (14)$$

For  $k \geq 3$  substituting  $t_1 = 0$ , generalized Lucas polynomials  $(G_{k,n}(t))$  and matrices  $D_{(k)}^\infty$  are reduced to following polynomials  $R_{k,n}^i(t)$ ; for  $n > 0$ ,  $1 \leq i \leq k$

$$R_{k,n}^i(t) = \sum_{j=2}^k t_j R_{k,n-j}^i(t) \quad (15)$$



with boundary conditions for  $1 - k \leq n \leq 0$ ,

$$R_{(k)} = [a_{k+n,i}] = R_{k,n}^i(t).$$

**Definition 2.6.** The polynomials  $R_{k,n}^i(t)$  derived (15) is called  $k$  sequences of generalized Perrin polynomials.

For  $k \geq 3$  and substituting  $t_1 = 0$  matrix  $D_{(k)}^\infty$  reduces to matrix  $R_{(k)}^\infty$ . Right hand column of  $R_{(k)}^\infty$  contains Generalized Perrin polynomials  $R_{k,n}(t)$ .  $i$ -th column of matrix  $R_{(k)}^\infty$  becomes  $i$ -th sequence of  $k$  sequences of generalized Perrin polynomials vice versa.

We note that matrix  $R_{(k)}^\infty$  for  $t_1 = 0$  in matrix  $D_{(k)}^\infty$  and  $R_{(k)}^\infty$  contains  $k$  sequence of Perrin polynomials  $R_{k,n}^i(t)$ . Let  $\widetilde{R}_n$  be generalized Perrin matrix is obtained by  $R_{(k)} \cdot (V_{(k)})^n$  as;

$$\widetilde{R}_n = R_{(k)} \cdot (V_{(k)})^n = \begin{bmatrix} R_{k,n-k+1}^1(t) & R_{k,n-k+1}^2(t) & \cdots & R_{k,n-k+1}^k(t) \\ \vdots & \vdots & \cdots & \vdots \\ R_{k,n-1}^1(t) & R_{k,n-1}^2(t) & \ddots & R_{k,n-1}^k(t) \\ R_{k,n}^1(t) & R_{k,n}^2(t) & \cdots & R_{k,n}^k(t) \end{bmatrix}. \quad (16)$$

**Example 2.7.** We give matrix  $R_{(k)}^\infty$  and  $k$  sequences of generalized Perrin polynomials for  $k = 3$  respectively,

$$R_{(3)}^\infty = \begin{bmatrix} -\frac{t_2^3}{t_3^2} + 3 & -\frac{t_2}{t_3} & \frac{t_2^2}{t_3^2} \\ \frac{t_2^2}{t_3} & 3 & -\frac{t_2}{t_3} \\ -t_2 & 0 & 3 \\ 3t_3 & 2t_2 & 0 \\ 0 & 3t_3 & 2t_2 \\ 2t_2t_3 & 2t_2^2 & 3t_3 \\ \vdots & \vdots & \vdots \end{bmatrix} \quad \text{and} \quad \begin{array}{c|c|c|c} n \setminus i & 1 & 2 & 3 \\ \hline -2 & -\frac{t_2^3}{t_3^2} + 3 & -\frac{t_2}{t_3} & \frac{t_2^2}{t_3^2} \\ \hline -1 & \frac{t_2^2}{t_3} & 3 & -\frac{t_2}{t_3} \\ \hline 0 & -t_2 & 0 & 3 \\ \hline 1 & 3t_3 & 2t_2 & 0 \\ \hline 2 & 0 & 3t_3 & 2t_2 \\ \hline 3 & 2t_2t_3 & 2t_2^2 & 3t_3 \\ \hline \vdots & \vdots & \vdots & \vdots \end{array}$$

Now we give four Corollary by using properties of Generalized Fibonacci and Lucas Numbers.

**Corollary 2.8.**  $tr(V_{(k)}^n) = R_{k,n}(t)$  for  $n \in \mathbb{Z}$ , where  $R_{k,n}(t)$  is the Generalized Perrin polynomials.

**Corollary 2.9.**  $V_{k,n}^i(t)$  be the  $k$  sequences of generalized Perrin and Van der Laan polynomials respectively, then

$$V_{k,n}^k(t) = V_{k,n-1}^{k-1}(t)$$

and

$$V_{k,n}^1(t) = t_k V_{k,n-1}^k(t)$$

for  $n \geq 1$ .

**Corollary 2.10.**  $R_{k,n}^i(t)$  be the  $k$  sequences of generalized Perrin polynomials respectively, then

$$R_{k,n}^k(t) = R_{k,n-1}^{k-1}(t)$$

and

$$R_{k,n}^1(t) = t_k R_{k,n-1}^k(t)$$

for  $n \geq 1$ .

**Corollary 2.11.**  $R_{k,n}^i(t)$  and  $V_{k,n}^i(t)$  are  $k$  sequences of generalized Perrin and Van der Laan polynomials respectively, then

$$\sum_{j=1}^k \frac{\partial R_{k,n}^k(t)}{\partial t_j} t_j = n V_{k,n}^k(t).$$

**Theorem 2.12.**  $R_{k,n}^i(t)$  and  $V_{k,n}^i(t)$  are  $k$  sequences of generalized Perrin and Van der Laan polynomials respectively, then

$$R_{k,n}^i(t) = (-t_{k-1})V_{k,n-k+1}^i(t) + \dots + (-t_2(k-2))V_{k,n-2}^i(t) + kV_{k,n}^i(t).$$

*Proof.* Using (13) and (16) we obtain

$$\begin{aligned}
\widetilde{R}_n &= R_{(k)} \widetilde{V}_n \\
\Rightarrow &\begin{bmatrix} R_{k,n-k+1}^1(t) & R_{k,n-k+1}^2(t) & \cdots & R_{k,n-k+1}^k(t) \\ \vdots & \vdots & \cdots & \vdots \\ R_{k,n-1}^1(t) & R_{k,n-1}^2(t) & \ddots & R_{k,n-1}^k(t) \\ R_{k,n}^1(t) & R_{k,n}^2(t) & \cdots & R_{k,n}^k(t) \end{bmatrix} \\
&= \begin{bmatrix} (-t_{k-1}, \dots, -t_2(k-2), 0, k) \cdot (V_{(k)})^{-(k)} \\ (-t_{k-1}, \dots, -t_2(k-2), 0, k) \cdot (V_{(k)})^{-(k-1)} \\ \vdots \\ (-t_{k-1}, \dots, -t_2(k-2), 0, k) \cdot (V_{(k)})^{-1} \\ (-t_{k-1}, \dots, -t_2(k-2), 0, k) \end{bmatrix} \\
&\begin{bmatrix} V_{k,n-k+1}^1(t) & V_{k,n-k+1}^2(t) & \cdots & V_{k,n-k+1}^k(t) \\ \vdots & \vdots & \cdots & \vdots \\ V_{k,n-1}^1(t) & V_{k,n-1}^2(t) & \ddots & V_{k,n-1}^k(t) \\ V_{k,n}^1(t) & V_{k,n}^2(t) & \cdots & V_{k,n}^k(t) \end{bmatrix}.
\end{aligned}$$

After use matrix multiplication we get

$$R_{k,n}^i(t) = (-t_{k-1})V_{k,n-k+1}^i(t) + \cdots + (-t_2(k-2))V_{k,n-1}^i(t) + kV_{k,n}^i(t).$$

□

**Example 2.13.** We obtain  $R_{4,5}^3(t)$  using Theorem (2.12)

$$\begin{aligned}
R_{4,5}^3(t) &= (-t_3)V_{4,5-4+1}^3(t) + (-t_2(k-2))V_{4,5-3+1}^3(t) + kV_{4,5}^3(t) \\
&= (-t_3)V_{4,2}^3(t) + (-t_2(4-2))V_{4,3}^3(t) + kV_{4,5}^3(t) \\
&= (-t_3)t_3 + (-2t_2)(t_4 + t_2^2) + 4(t_3^2 + t_2^3) = 6t_2t_4 + 2t_2^3 + 3t_3^2.
\end{aligned}$$

**Theorem 2.14.** Let  $V_{k,n}^i(t)$  be the  $k$  sequences of generalized Van der Laan polynomials, then

$$V_{k,n+m}^i(t) = \sum_{j=1}^k V_{k,m}^j(t) V_{k,n-k+j}^i(t).$$

*Proof.* We know that  $\widetilde{V}_n = (V_{(k)})^n$ . We may rewrite it as

$$\begin{aligned} (V_{(k)})^{n+1} &= (V_{(k)})^n (V_{(k)}) = (V_{(k)}) (V_{(k)})^n \\ \Rightarrow \widetilde{V}_{n+1} &= \widetilde{V}_n \widetilde{V}_1 = \widetilde{V}_1 \widetilde{V}_n \end{aligned}$$

and inductively

$$\widetilde{V}_{n+m} = \widetilde{V}_n \widetilde{V}_m = \widetilde{V}_m \widetilde{V}_n. \quad (17)$$

Consequently, any element of  $\widetilde{V}_{n+m}$  is the product of a row of  $\widetilde{V}_n$  and a column of  $\widetilde{V}_m$ ; that is

$$V_{k,n+m}^i(t) = \sum_{j=1}^k V_{k,m}^j(t) V_{k,n-k+j}^i(t).$$

□

**Corollary 2.15.** *Taking  $n = m$  in (17) we obtain  $(\widetilde{V}_n)^2 = \widetilde{V}_n \widetilde{V}_n = \widetilde{V}_{n+n} = \widetilde{V}_{2n}$ .*

**Theorem 2.16.**  *$R_{k,n}^i(t)$  and  $V_{k,n}^i(t)$  are  $k$  sequences of generalized Perrin and Van der Laan polynomials respectively, then*

$$R_{k,n}^i(t) = kt_k V_{k,n-k}^i(t) + \cdots + 3t_3 V_{k,n-3}^i(t) + 2t_2 V_{k,n-2}^i(t).$$

*Proof.*

$$\begin{aligned}
\widetilde{R}_n &= R_{(k)} \widetilde{V}_n = R_{(k)} \widetilde{V}_1 \widetilde{V}_{n-1} \\
&\Rightarrow \begin{bmatrix} R_{k,n-k+1}^1(t) & R_{k,n-k+1}^2(t) & \cdots & R_{k,n-k+1}^k(t) \\ \vdots & \vdots & \cdots & \vdots \\ R_{k,n-1}^1(t) & R_{k,n-1}^2(t) & \ddots & R_{k,n-1}^k(t) \\ R_{k,n}^1(t) & R_{k,n}^2(t) & \cdots & R_{k,n}^k(t) \end{bmatrix} \\
&= \begin{bmatrix} (-t_{k-1}, \dots, -t_2(k-2), 0, k) \cdot (V_{(k)})^{-(k)} \\ (-t_{k-1}, \dots, -t_2(k-2), 0, k) \cdot (V_{(k)})^{-(k-1)} \\ \vdots \\ (-t_{k-1}, \dots, -t_2(k-2), 0, k) \cdot (V_{(k)})^{-1} \\ (-t_{k-1}, \dots, -t_2(k-2), 0, k) \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ t_k & t_{k-1} & t_{k-2} & \cdots & 0 \end{bmatrix} \\
&\begin{bmatrix} V_{k,n-k}^1(t) & V_{k,n-k}^2(t) & \cdots & V_{k,n-k}^k(t) \\ \vdots & \vdots & \cdots & \vdots \\ V_{k,n-2}^1(t) & V_{k,n-2}^2(t) & \ddots & V_{k,n-2}^k(t) \\ V_{k,n-1}^1(t) & V_{k,n-1}^2(t) & \cdots & V_{k,n-1}^k(t) \end{bmatrix} \\
&= \begin{bmatrix} (-t_{k-1}, \dots, -t_2(k-2), 0, k) \cdot (V_{(k)})^{-(k-1)} \\ (-t_{k-1}, \dots, -t_2(k-2), 0, k) \cdot (V_{(k)})^{-(k-2)} \\ \vdots \\ (-t_{k-1}, \dots, -t_2(k-2), 0, k) \\ (kt_k, \dots, 3t_3, 2t_2, 0) \end{bmatrix} \\
&\begin{bmatrix} V_{k,n-k}^1(t) & V_{k,n-k}^2(t) & \cdots & V_{k,n-k}^k(t) \\ \vdots & \vdots & \cdots & \vdots \\ V_{k,n-2}^1(t) & V_{k,n-2}^2(t) & \ddots & V_{k,n-2}^k(t) \\ V_{k,n-1}^1(t) & V_{k,n-1}^2(t) & \cdots & V_{k,n-1}^k(t) \end{bmatrix}.
\end{aligned}$$

Using matrix multiplication we obtain

$$R_{k,n}^i(t) = kt_k V_{k,n-k}^i(t) + \cdots + 3t_3 V_{k,n-3}^i(t) + 2t_2 V_{k,n-2}^i(t).$$

□

### 3. Generalized order- $k$ Van der Laan and Perrin numbers

In this section we define generalized order- $k$  Van der Laan numbers  $v_{k,n}$  and  $k$  sequences of the generalized order- $k$  Van der Laan numbers  $v_{k,n}^i$  by the help of  $k$  sequences of Generalized Van der Laan Polynomials. In addition we

define generalized order- $k$  Perrin numbers  $r_{k,n}$  and  $k$  sequences of the generalized order- $k$  Perrin numbers  $r_{k,n}^i$  by the help of  $k$  sequences of Generalized Perrin Polynomials.

**Definition 3.1.** For  $t_s = 1, 2 \leq s \leq k$ , the generalized Van der Laan polynomial  $V_{k,n}(t)$  and  $V_{(k)}^\infty$  together are reduced to

$$v_{k,n} = \sum_{j=2}^k v_{k,n-j}$$

with boundary conditions

$$v_{k,1-k} = v_{k,2-k} = \dots = v_{k,-2} = 0, \quad v_{k,-1} = 1 \text{ and } v_{k,0} = 0,$$

which is called generalized order- $k$  Van der Laan numbers(GOkV).

When  $k = 3$ , it is reduced to ordinary Van der Laan numbers.

**Definition 3.2.** For  $t_s = 1, 2 \leq s \leq k$ ,  $V_{k,n}^i(t)$  can be written explicitly as

$$v_{k,n}^i = \sum_{j=2}^k v_{k,n-j}^i \quad (18)$$

for  $n > 0$  and  $1 \leq i \leq k$ , with boundary conditions

$$v_{k,n}^i = \begin{cases} 1 & \text{if } i - n = k, \\ 0 & \text{otherwise} \end{cases}$$

for  $1 - k \leq n \leq 0$ , where  $v_{k,n}^i$  is the  $n$ -th term of  $i$ -th sequence. This generalization is called  $k$  sequences of the generalized order- $k$  Van der Laan numbers( $kSOkV$ ).

When  $i = k = 3$ , we obtain ordinary Van der Laan numbers and for any integer  $k$ ,  $v_{k,n-1}^k = v_{k,n}$ .

**Example 3.3.** Substituting  $k = 3$  and  $i = 2$  we obtain the generalized order-3 Van der Laan sequence as;

$$v_{3,-2}^2 = 0, \quad v_{3,-1}^2 = 1, \quad v_{3,0}^2 = 0, \quad v_{3,1}^2 = 1, \quad v_{3,2}^2 = 1, \quad v_{3,3}^2 = 1, \quad v_{3,4}^2 = 2, \dots$$

We firstly give some properties of  $k$  sequences of the generalized order- $k$  Van der Laan numbers( $kSOkV$ ) using properties of  $k$  sequences of generalized Van der Laan polynomials( $V_{k,n}^i(t)$ ).

**Corollary 3.4.** *Matrix multiplication and (18) can be used to obtain*

$$V_n^\sim = A_1^n$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 0 \end{bmatrix}_{k \times k} = \begin{bmatrix} 0 & & & \\ 0 & & I & \\ 0 & & & \\ 1 & \dots & 1 & 0 \end{bmatrix}_{k \times k} \quad (19)$$

where  $I$  is  $(k-1) \times (k-1)$  identity matrix and  $V_n^\sim$  is a matrix as;

$$V_n^\sim = \begin{bmatrix} v_{k,n-k+1}^1 & v_{k,n-k+1}^2 & \dots & v_{k,n-k+1}^k \\ \vdots & \vdots & \dots & \vdots \\ v_{k,n-1}^1 & v_{k,n-1}^2 & \dots & v_{k,n-1}^k \\ v_{k,n}^1 & v_{k,n}^2 & \dots & v_{k,n}^k \end{bmatrix} \quad (20)$$

which is contained by  $k \times k$  block of  $V_{(k)}^\infty$  for  $t_i = 1, 2 \leq i \leq k$ .

*Proof.* It is clear that  $V_1^\sim = A_1$  and  $V_{n+1}^\sim = A_1 V_n^\sim$  by (18). So by induction, we have  $V_n^\sim = A_1^n$ .  $\square$

**Corollary 3.5.** *Let  $V_n^\sim$  be as in (20). Then*

$$\det V_n^\sim = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ (-1)^n & \text{if } k \text{ is even.} \end{cases}$$

*Proof.* Obvious from (8).  $\square$

**Corollary 3.6.** *Let  $v_{k,n}^i$  be the  $i$ -th sequences of  $k$  sequences of generalized order- $k$  Van der Laan Numbers, for  $1 \leq i \leq k$ . Then, for all positive integers  $n$  and  $m$*

$$v_{k,n+m}^i = \sum_{j=1}^k v_{k,m}^j v_{k,n-k+j}^i.$$

*Proof.* Obvious from Theorem (2.14).  $\square$

**Corollary 3.7.** *Let  $v_{k,n}^i$  be the  $i$ -th sequences of  $kSOkV$ . Then, for  $n > 1 - k$ ,*

$$v_{k,n}^1 = v_{k,n-1}^k = v_{k,n-2}^{k-1}. \quad (21)$$

*Proof.* It is obvious from Definition (3.2) that this sequence are equal with index iteration.  $\square$

**Lemma 3.8.** *Let  $v_{k,n}^i$  be the  $i$ -th sequences of  $kSOkV$  then*

$$v_{k,n}^i = v_{k,n}^{i-1} + v_{k,n-i}^k \quad (22)$$

*for  $n > 1 - k + i$  and  $1 < i \leq k$ .*

*Proof.* Assume for  $n > 1 - k + i$ ,  $v_{k,n}^i - v_{k,n}^{i-1} = t_n$  and show  $t_n = v_{k,n-i}^k$ .

First we obtain initial conditions for  $t_n$  by using initial conditions of  $i$ -th and  $(i - 1)$ -th sequences of  $kSOkV$  simultaneously as follows;

$n \setminus i$	$v_{k,n}^i$	$v_{k,n}^{i-1}$	$t_n = v_{k,n}^i - v_{k,n}^{i-1}$
$1 - k$	0	0	0
$2 - k$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$i - k - 2$	0	0	0
$i - k - 1$	0	1	-1
$i - k$	1	0	1
$i - k + 1$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	0	0	0

Since initial conditions of  $t_n$  are equal to initial condition of  $v_{k,n}^k$  with index iteration. Then we have

$$t_n = v_{k,n-i}^k.$$

$\square$

We give the following Theorem by using generalization of MacHenry in [10].



**Theorem 3.9.** Let  $v_{k,n}^i$  be the  $i$ -th sequences of  $kSOkV$ , then for  $n \geq 1$  and  $1 \leq i \leq k$

$$v_{k,n}^i = v_{k,n-1}^k + v_{k,n-2}^k + \cdots + v_{k,n-i}^k = \sum_{m=1}^i v_{k,n-m}^k.$$

*Proof.* Writting equalty (22) recursively we have

$$\begin{aligned} v_{k,n}^{i+1} - v_{k,n}^i &= v_{k,n-i-1}^k \\ v_{k,n}^{i+2} - v_{k,n}^{i+1} &= v_{k,n-i-2}^k \\ &\vdots \\ v_{k,n}^{k-2} - v_{k,n}^{k-3} &= v_{k,n-k+2}^k \\ v_{k,n}^{k-1} - v_{k,n}^{k-2} &= v_{k,n-k+1}^k \end{aligned}$$

and adding these equations side by side we obtain

$$v_{k,n}^{k-1} - v_{k,n}^i = v_{k,n-k+1}^k + v_{k,n-k+2}^k + \cdots + v_{k,n-i-2}^k + v_{k,n-i-1}^k.$$

And using the equation  $v_{k,n}^{k-1} = v_{k,n+1}^k$  and Definition (3.4) we obtain

$$\begin{aligned} v_{k,n}^i &= v_{k,n+1}^k - (v_{k,n-k+1}^k + v_{k,n-k+2}^k + \cdots + v_{k,n-i-2}^k + v_{k,n-i-1}^k) \\ &= v_{k,n-1}^k + v_{k,n-2}^k + \cdots + v_{k,n-k+1}^k \\ &\quad - (v_{k,n-k+1}^k + v_{k,n-k+2}^k + \cdots + v_{k,n-i-2}^k + v_{k,n-i-1}^k) \\ &= v_{k,n-1}^k + v_{k,n-2}^k + \cdots + v_{k,n-i}^k. \end{aligned}$$

□

Now we initiate to the generalized Perrin numbers.

**Definition 3.10.** For  $t_s = 1$ ,  $2 \leq s \leq k$ , the generalized Perrin polynomial  $R_{k,n}(t)$  and matrix  $R_{(k)}^\infty$  together are reduced to

$$r_{k,n} = \sum_{j=2}^k r_{k,n-j} \tag{23}$$

with boundary conditions

$$r_{k,1-k} = (k-2), \quad r_{k,2-k} = \cdots = r_{k,-2} = r_{k,-1} = -1 \text{ and } r_{k,0} = k,$$

which is called generalized order- $k$  Perrin numbers(GOkR).

When  $k = 3$ , it is reduced to ordinary Perrin numbers;  $(1, (-1), 3, 0, 2, 3, 2, 5, 5, 7, \dots)$  with iterating index by two.

We rewrite matrix (14) for  $t_s = 1, 2 \leq s \leq k$  we obtain

$$R_{(k1)} = [a_{n,i}]_{k \times k} = \begin{bmatrix} ((-1), (-2), \dots, (k-2), 0, k) \cdot (A_1)^{-(k-1)} \\ ((-1), (-2), \dots, (k-2), 0, k) \cdot (A_1)^{-(k-2)} \\ \vdots \\ ((-1), (-2), \dots, (k-2), 0, k) \cdot (A_1)^{-1} \\ ((-1), (-2), \dots, (k-2), 0, k) \end{bmatrix}.$$

**Definition 3.11.** For  $t_s = 1, 2 \leq s \leq k, R_{k,n}^i(t)$  can be written explicitly as

$$r_{k,n}^i = \sum_{j=2}^k r_{k,n-j}^i$$

for  $n > 0$  and  $1 \leq i \leq k$ , with boundary conditions

$$r_{k,n}^i = [a_{k+n,i}]_{k \times k} = R_{(k1)}$$

for  $1 - k \leq n \leq 0$ , where  $r_{k,n}^i$  is the  $n$ -th term of  $i$ -th sequence. This generalization is called  $k$  sequences of the generalized order- $k$  Perrin numbers( $kSOkR$ ).

When  $i = k = 3$ , we obtain ordinary Perrin numbers and for any integer  $k \geq 3, r_{k,n}^k = r_{k,n}$ .

**Corollary 3.12.** Let  $r_{k,n}^i$  and  $v_{k,n}^i$  be  $k$  sequences of generalized Perrin and Van der Laan numbers respectively, then

$$r_{k,n}^i = kv_{k,n}^i - (v_{k,n-k+1}^i + \dots + (k-2)v_{k,n-2}^i).$$

**Corollary 3.13.** Let  $r_{k,n}^i$  and  $v_{k,n}^i$  be  $k$  sequences of generalized Perrin and Van der Laan numbers respectively, then

$$r_{k,n}^i(t) = kv_{k,n-k}^i + \dots + 3v_{k,n-3}^i + 2v_{k,n-2}^i.$$

**Conclusion 3.14.** There are a lot of studies on Fibonacci and Lucas numbers and on their generalizations. In this paper we showed that these studies can be transferred to the Van der Laan and Perrin numbers. Since our definition of these number are polynomial based, it has great amount of application area.

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